§ 14.4 Chain rule:

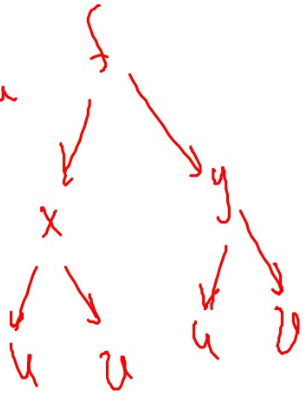
$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$$

$f(x,y)$; $x(u,v)$; $y(u,v)$

$$\frac{\partial f}{\partial u} = f_u = f_x x_u + f_y y_u$$

or

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$



$f(x)$; $x(t)$

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

$$f \to x \to t$$

$$y = (x^2+1)^3$$

$$\frac{dy}{dx} = 3(x^2+1)^2 \cdot (2x)$$

$$F(x,y) = K$$

$$\begin{tikzcd} & F \arrow[dl] \arrow[dr] & \\ x \arrow[dr] & & y \arrow[dl] \\ & x & \end{tikzcd}$$

$$\frac{dF}{dx} = \frac{dK}{dx}$$

$$F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = 0$$

$$F_x + F_y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$F(x,y,z) = c$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

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$$x^2 + xy + y^2 - 7 = 0 \qquad \frac{dy}{dx}(1,2) = ?$$

let $F(x,y) = x^2 + xy + y^2 - 7$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+y}{x+2y}$$

$$\frac{dy}{dx}(1,2) = -\frac{4}{5}$$

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let $\boxed{F(x,y,z) = z^3 - xy + yz + y^3}$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-y}{3z^2+y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-x+z+3y^2}{3z^2+y}$$

$$\frac{\partial z}{\partial x}(1,1,1) = \frac{1}{4}$$

$$\frac{\partial z}{\partial y}(1,1,1) = -\frac{3}{4}$$

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$$f(u, v, w)$$

$$u = x - y$$
$$v = y - z$$
$$w = z - x$$

$$\overset{??}{\Longrightarrow} \quad f_x + f_y + f_z = 0$$

$u_x = 1$

$u_y = -1$

$\ldots$

$$f_x = f_u u_x + f_v v_x + f_w w_x = f_u + 0 - f_w$$

$$\frac{\partial f}{\partial y} = f_y = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = -f_u + f_v + 0$$

$$f_z = f_u u_z + f_v v_z + f_w w_z = -f_v + f_w$$

$$\therefore \quad f_x + f_y + f_z = 0.$$

E4 $w = f(x,y)$; $x = r\cos\theta$; $y = r\sin\theta$

a) $$\boxed{\frac{\partial w}{\partial r} = f_x \cos\theta + f_y \sin\theta}$$

$$\frac{\partial w}{\partial \theta} = -r f_x \sin\theta + r f_y \cos\theta$$

$$\boxed{\frac{1}{r}\frac{\partial w}{\partial \theta} = -f_x \sin\theta + f_y \cos\theta}$$ if $r \neq 0$

c) $$(f_x)^2 + (f_y)^2 \overset{??}{=} \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2$$

use part a

square both sides and simplify

b) $$f_x = \frac{\begin{vmatrix} \partial w/\partial r & \sin\theta \\ \frac{1}{r}\frac{\partial w}{\partial \theta} & \cos\theta \end{vmatrix}}{\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}} \quad (=1)$$

$$= \frac{\partial w}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial w}{\partial \theta}\sin\theta.$$

$$f_y = \frac{\begin{vmatrix} \cos\theta & \partial w/\partial r \\ -\sin\theta & \frac{1}{r}\frac{\partial w}{\partial \theta} \end{vmatrix}}{1}$$

$$= \frac{1}{r}\cos\theta\frac{\partial w}{\partial \theta} + \sin\theta\frac{\partial w}{\partial r}.$$

例 $w=f(u,v)$ s.t. $f_{uu}+f_{vv}=0$

$$\left.\begin{aligned} &w=f(u,v) \text{ s.t. } f_{uu}+f_{vv}=0 \\ &u=\frac{x^2}{2}-\frac{y^2}{2} \ ; \ v=xy \end{aligned}\right\} \overset{??}{\Longrightarrow} w_{xx}+w_{yy}=0$$

$$w_x = f_u u_x + f_v v_x = x f_u + y f_v$$

$$w_{xx} = f_u + x\left(f_{uu}u_x + f_{uv}v_x\right) + y\left(f_{vu}u_x + f_{vv}v_x\right)$$

$$w_{xx} = f_u + x^2 f_{uu} + xy f_{uv} + xy f_{vu} + y^2 f_{vv}$$

$$w_y = -y f_u + x f_v$$

$$w_{yy} = -f_u - y\left(f_{uu}u_y + f_{uv}v_y\right) + x\left(f_{vu}u_y + f_{vv}v_y\right)$$

$$w_{yy} = -f_u + y^2 f_{uu} - xy f_{uv} - xy f_{vu} + x^2 f_{vv}$$

$$\begin{tikzcd} & f \equiv w \arrow[dl] \arrow[dr] & \\ u \arrow[dr] & & v \arrow[dl] \\ & (x,y) & \end{tikzcd}$$

$$\begin{tikzcd} & f_u \arrow[dl] \arrow[dr] & \\ u \arrow[dr] & & v \arrow[dl] \\ & (x,y) & \end{tikzcd}$$

$$\begin{tikzcd} & f_v \arrow[dl] \arrow[dr] & \\ u \arrow[dr] & & v \arrow[dl] \\ & (x,y) & \end{tikzcd}$$